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A dual to Lyapunov's stability theorem

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Abstract

Lyapunov's second theorem is a standard tool for stability analysis of ordinary differential equations. Here we introduce a theorem which can be viewed as a dual to Lyapunov's result. From existence of a scalar function satisfying certain inequalities it follows that "almost all trajectories" of the system tend to zero. The scalar function has a physical interpretation as the stationary density of a substance that is generated in all points of the state space and flows along the system trajectories. If the stationary density is bounded everywhere except at a singularity in the origin, then almost all trajectories tend towards the origin. The weaker notion of stability allows for applications also in situations where Lyapunov's theorem cannot be used. Moreover, the new criterion has a striking convexity property related to control synthesis.

Keywords: Stabilization; Nonlinear systems; Convex duality

1. Introduction

Lyapunov's second theorem has long been recognized as one of the most fundamental tools for analysis and synthesis of nonlinear systems. The importance of the criterion stems from the fact that it allows stability of a system to be verified without solving the differential equation explicitly.

The original work of Lyapunov in the late 19th century was devoted to problems from astronomy and fluid mechanics. In the 1950s, it was applied by Chetayev to aeronautical stability problems and by Lur'e and Letov for nonlinear control problems. The ideas were promoted in the 1960s by Kalman, Lefschetz and La Salle and have found widespread applications since then [6,12,22,11,9,15].

Lyapunov functions play a role similar to potential functions and energy functions. Moreover, when asymptotic stability of an equilibrium has been proved using Lyapunov's theorem, input–output stability can often be proved using the Lyapunov function as a "storage function" [25].

It is surprising to find that Lyapunov's theorem has a close relative, presented here as Theorem 1, that has been neglected until present date. This is even more striking as the relationship between the two theorems is analogous to the duality that has been used since 1940s for closely related problems in calculus of variations [10,26,24,21]. The new result is similar to the Bendixson–Dulac theorem for two-dimensional systems [1], a result which also has other generalizations to higher dimensions [23,17].

The outline of the paper is as follows. The new convergence criterion is presented in Section 2 and followed by a few examples. The relationship between Lyapunov functions and density functions is discussed

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in Section 3 and the duality is explained from an intuitive viewpoint in Section 4.

In Section 5 we make a connection to more recent work on feedback control based on Lyapunov functions [3,8,13,14]. Some of the difficulties in stabilization of nonlinear systems can be associated with the fact that the set of "control Lyapunov functions" has a difficult structure. For some systems, it is not even connected. It is therefore interesting to note that the corresponding set for the new convergence criterion is convex.

The full proof of the main theorem is not given until Section 6. The argument is based on a theorem for abstract measures which is also useful in the proof of many other related results.

The notation

$$\nabla V = \left[\frac{\partial V}{\partial x_1} \cdots \frac{\partial V}{\partial x_n}\right], \quad V : \mathbb{R}^n \to \mathbb{R},$$

$$\nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n}, \quad f : \mathbb{R}^n \to \mathbb{R}^n$$

will be used throughout the paper.

2. The main result

Theorem 1. *Given the equation $\dot{x}(t) = f(x(t))$, where $f \in \mathbb{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and $f(0) = 0$, suppose there exists a non-negative $\rho \in \mathbb{C}^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ such that $\rho(x) f(x)/|x|$ is integrable on $\{x \in \mathbb{R}^n\colon |x| \geqslant 1\}$ and*

$$[\nabla \cdot (f\rho)](x) > 0 \quad \text{for almost all } x. \tag{1}$$

Then, for almost all initial states $x(0)$ the trajectory $x(t)$ exists for $t \in [0, \infty)$ and tends to zero as $t \to \infty$. Moreover, if the equilibrium $x = 0$ is stable, then the conclusion remains valid even if ρ takes negative values.

Proof (*Second statement*). Here it is assumed that $x = 0$ is a stable equilibrium, while ρ may take negative values. The proof for the other case is given in Section 6.

Rather than exploiting that $f \in \mathbb{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, we will actually prove the result under the weaker condition that $f \in \mathbb{C}^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$ and $f(x)/|x|$ is bounded near $x=0$. Given any $x_0 \in \mathbb{R}^n$, let $\phi_t(x_0)$ for $t \geqslant 0$ be the solution $x(t)$ of $\dot{x}(t) = f(x(t))$, $x(0) = x_0$. Assume first that ρ is integrable on $\{x \in \mathbb{R}^n\colon |x| \geqslant 1\}$ and $|f(x)|/|x|$ is bounded. Then ϕ_t is well defined for all t. Given $r > 0$, define

$$Z = \bigcap_{l=1}^{\infty} \{x_0\colon |\phi_t(x_0)| > r \text{ for some } t > l\}. \tag{2}$$

Notice that Z contains all trajectories with $\limsup_{t\to\infty} |x(t)| > r$. The set Z, being the intersection of a countable number of open sets, is measurable. Moreover, $\phi_t(Z) = \{\phi_t(x) \mid x \in Z\}$ is equal to Z for every t. By stability of the equilibrium $x = 0$, there is a positive lower bound ε on the norm of the elements in Z, so Lemma A.1 with $D = \{x\colon |x| > \varepsilon\}$ gives

$$\begin{aligned} 0 &= \int_{\phi_t(Z)} \rho(x)\,\mathrm{d}x - \int_Z \rho(z)\,\mathrm{d}z \\ &= \int_0^t \int_{\phi_\tau(Z)} [\nabla \cdot (f\rho)](x)\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{3}$$

By assumption (1), this implies that Z has measure zero. Consequently, $\limsup_{t\to\infty} |x(t)| \leqslant r$ for almost all trajectories. As r was chosen arbitrarily, this proves that $\lim_{t\to\infty} |x(t)| = 0$ for almost all trajectories.

When $|f(x)|/|x|$ is unbounded, there may not exist any non-zero t such that $\phi_t(z)$ is well defined for all z. We then introduce

$$\rho_0(x) = \left[\frac{\mathrm{e}^{-|x|}}{1 + |\rho(x)|^2} + \frac{|f(x)|^2}{|x|^2}\right]^{1/2} \rho(x),$$

$$f_0(x) = \frac{f(x)\rho(x)}{\rho_0(x)}.$$

Then $|f_0(x)|/|x|$ is bounded and ρ_0 is integrable on $\{x \in \mathbb{R}^n\colon |x| \geqslant 1\}$, so the argument above can be applied to f_0 together with ρ_0 to prove that $\lim_{\tau\to\infty} |y(\tau)| = 0$ for almost all trajectories of the system $\mathrm{d}y/\mathrm{d}\tau = f_0(y(\tau))$. However, modulo a transformation of the time axis

$$t = \int_0^\tau \frac{\rho(y(s))}{\rho_0(y(s))}\,\mathrm{d}s$$

the trajectories are identical: $x(t) = y(\tau)$. This, together with the boundedness of $f(x)/|x|$ near $x = 0$, also shows that $x(t)$ exists for $t \in [0, \infty)$ and tends to zero as $t \to \infty$ provided that $\lim_{\tau\to\infty} |y(\tau)| = 0$. Hence the proof of the second statement in Theorem 1 is complete. □

Example 1. For scalar x, define

$$f(x) = x, \quad \rho(x) = -\frac{1}{x^4}.$$

Then $[\nabla \cdot (f\rho)](x) = 3/x^4 > 0$, so all conditions of Theorem 1 hold except for non-negativity of ρ and stability of $x = 0$.

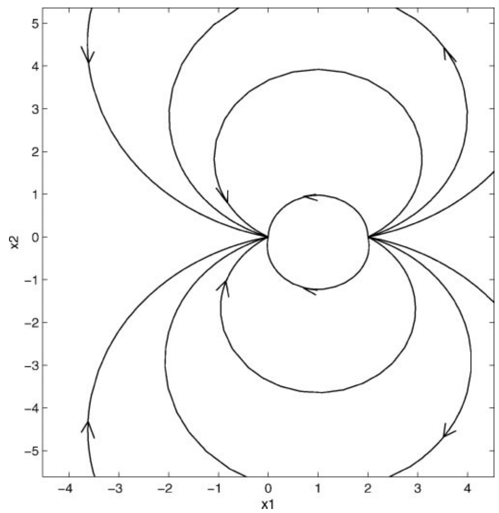

Fig. 1. Phase plane plot for Example 3.

Example 2. With

$$f(x) = (x^2 - 1)x, \quad \rho(x) = \frac{1}{x^2}$$

we have $[\nabla \cdot (f\rho)](x) = 1 + x^{-2} > 0$, so all conditions of Theorem 1 hold except for the integrability of $f\rho/|x|$. In this case, all trajectories starting outside the interval $[-1, 1]$ have finite escape time.

Example 3. The system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_1^2 - x_2^2 \\ -2x_2 + 2x_1x_2 \end{bmatrix}$$

has two equilibria $(0, 0)$ and $(2, 0)$. See Fig. 1. Let $f(x)$ be the right-hand side and let $\rho(x) = |x|^{-\alpha}$. Then

$$\begin{aligned} [\nabla \cdot (f\rho)](x) &= \nabla\rho \cdot f + \rho(\nabla \cdot f) \\ &= -\alpha|x|^{-\alpha-2}x^{\mathrm{T}}f + |x|^{-\alpha}(4x_1 - 4) \\ &= -\alpha|x|^{-\alpha-2}(x_1 - 2)|x|^2 + |x|^{-\alpha}(4x_1 - 4) \\ &= |x|^{-\alpha}[(4 - \alpha)x_1 + 2\alpha - 4]. \end{aligned}$$

With $\alpha = 4$ all conditions of Theorem 1 hold, so almost all trajectories tend to $(0, 0)$ as $t \to \infty$. The exceptional trajectories turn out to be those that start with $x_1 \geqslant 2$, $x_2 = 0$.

Example 4. The system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_1^2 - x_2^2 \\ -6x_2 + 2x_1x_2 \end{bmatrix} \tag{4}$$

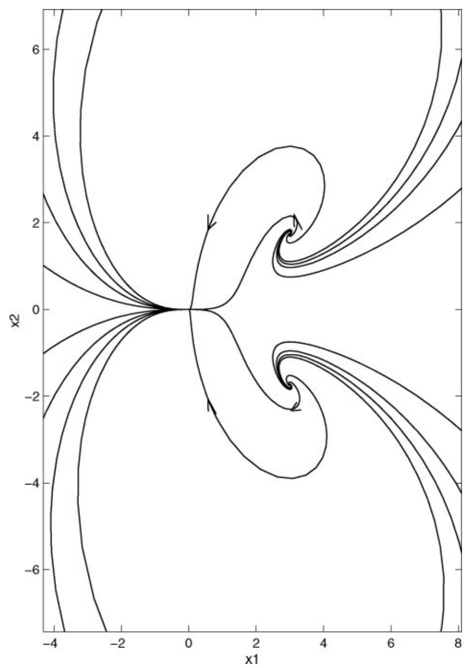

Fig. 2. Phase plane plot for Example 4.

has four equilibria $(0, 0)$, $(2, 0)$ and $(3, \pm\sqrt{3})$. See Fig. 2. In this case, $\rho(x) = |x|^{-4}$ gives

$$\begin{aligned} [\nabla \cdot (f\rho)](x) &= -4|x|^{-6}x^{\mathrm{T}}f + |x|^{-4}(4x_1 - 8) \\ &= -4|x|^{-6}[(x_1 - 2)|x|^2 - 4x_2^2] + |x|^{-4}(4x_1 - 8) \\ &= 16x_2^2|x|^{-6} \end{aligned}$$

so again Theorem 1 shows that almost all trajectories tend to $(0, 0)$ as $t \to \infty$. The exceptional trajectories are the three unstable equilibria, the axis $x_2 = 0$, $x_1 \geqslant 2$ and the stable manifold of the equilibrium $(2, 0)$, that spirals out from the equilibria $(3, \pm\sqrt{3})$.

3. Relation to Lyapunov functions

The fact that Lyapunov's theorem has a stronger implication than the convergence criterion of Theorem 1, suggests the possibility to derive a density function ρ from a Lyapunov function V. This can generally be done in the following way.

Proposition 1. *Let $V(x) > 0$ for $x \neq 0$ and*

$$\nabla V \cdot f < \alpha^{-1}(\nabla \cdot f)V \quad \textit{for almost all } x$$

for some $\alpha > 0$. Then $\rho(x) = V(x)^{-\alpha}$ satisfies the condition (1).

In particular, if P is a positive-definite matrix satisfying

$$A^{\mathrm{T}}P + PA < (\alpha^{-1}\operatorname{trace} A)P,$$

then $\rho(x) = (x^{\mathrm{T}}Px)^{-\alpha}$ *satisfies condition* (1) *for the system* $\dot{x} = Ax$.

Proof. With $\rho(x) = V(x)^{-\alpha}$, we get

$$\begin{aligned}\nabla \cdot (f\rho) &= (\nabla \cdot f)\rho + \nabla\rho \cdot f\\ &= (\nabla \cdot f)V^{-\alpha} - \alpha V^{-(\alpha+1)}\nabla V \cdot f\\ &= \alpha V^{-(\alpha+1)}[\alpha^{-1}(\nabla \cdot f)V - \nabla V \cdot f]\\ &> 0.\end{aligned}$$

With $V(x)=x^{\mathrm{T}}Px$ and $f(x)=Ax$ the second statement follows since

$$\nabla V \cdot f = x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x$$
$$\nabla \cdot f = \operatorname{trace} A. \qquad \square$$

Transfer in the opposite direction, from density function to Lyapunov function, is generally not possible. The simple reason is that a density function may exist even if the system is not globally asymptotically stable. This was the situation in Examples 3 and 4. However, with an additional assumption that $\nabla \cdot f \leqslant 0$, the following construction can be used.

Proposition 2. *Suppose for* $x \neq 0$ *that*

$$\nabla \cdot (f\rho) > 0, \quad \nabla \cdot f \leqslant 0, \quad \rho > 0.$$

Then $V(x) = \rho(x)^{-1}$ *satisfies* $\nabla V \cdot f < 0$.

Proof.

$$\begin{aligned}\nabla V \cdot f &= -\rho^{-2}\nabla\rho \cdot f\\ &= -\rho^{-2}[\nabla \cdot (f\rho) - (\nabla \cdot f)\rho] < 0. \qquad \square\end{aligned}$$

4. A viewpoint of duality

It is well known that Lyapunov functions are closely related to cost functions in optimal control. In fact, one way to interpret a Lyapunov function V for the globally stable dynamical system $\dot{x} = f(x)$ is to view $V(x_0)$ as the "cost to go" from the initial state x_0 to the equilibrium. The purpose of this section is to show that the cost function V and the density function ρ appear as duals to each other in a linear programming formulation of optimal control.

Let as first consider the discrete transportation problem illustrated in Fig. 3. Such problems have been studied extensively since 1940s [7,5]. Some product is produced with unit rate in each of the three nodes 1–3 and is consumed in node 0. The cost for shipping the product between node i and j is given by the number l_{ij}. It is well known that the minimal total transportation cost can be found by solving the following linear programming problem:

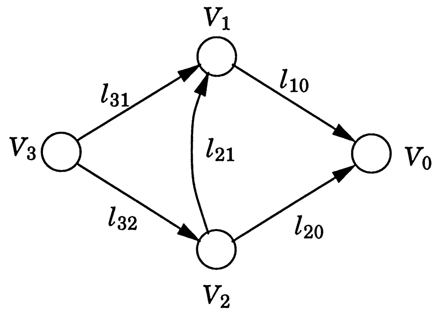

Fig. 3. The products produced in nodes 1–3 should be transported to the consumer in node 0 while minimizing the transportation cost.

$$\begin{array}{ll}\text{maximize} & V_1 + V_2 + V_3 - 3V_0\\ \text{subject to} & V_3 - V_1 \leqslant l_{31},\\ & V_3 - V_2 \leqslant l_{32},\\ & \vdots\\ & V_2 - V_0 \leqslant l_{20}.\end{array}$$

Note that there is one variable V_i for each node and one inequality constraint for each path connecting two nodes. For every solution to the inequality constraints, the number $V_i - V_0$ provides a lower bound on the cost for shipping products with unit rate from node i to node 0. The expression $V_1 + V_2 + V_3 - 3V_0$ therefore gives a lower bound on the total transportation cost.

A dual linear programming problem can be stated as follows:

$$\begin{array}{ll}\text{minimize} & l_{31}\rho_{31} + l_{32}\rho_{32} + l_{21}\rho_{21} + l_{10}\rho_{10} + l_{20}\rho_{20}\\ \text{subject to} & \rho_{31}, \ldots, \rho_{20} \geqslant 0,\\ & \rho_{31} + \rho_{32} \geqslant 1,\\ & -\rho_{31} - \rho_{21} + \rho_{10} \geqslant 1,\\ & -\rho_{32} + \rho_{21} + \rho_{20} \geqslant 1.\end{array}$$

For each path connecting two nodes, the variable ρ_{ij} can be interpreted as the transportation density from node i to node j. There is one constraint for each node stating that the total production in this node is at least as big as the assigned value.

The relation to Lyapunov functions and density functions is clear. The solution V to the primal linear programming problem is the optimal "cost to go". It is decreasing along the optimal transportation paths and serves a Lyapunov function for the optimal transportation dynamics. The solution to the dual linear programming problem is instead analogous to the function ρ appearing in Theorem 1.

The discrete optimization problem discussed so far has several continuous analogs [10,26,24,21]. For the sake of brevity, we restrict our attention to systems of the form $\dot{x}(t)=f(x(t)), x(0)=x_0$, i.e. a fixed controller. Consider a production rate $\psi(x)\geqslant 0$ and a transportation cost $l(x)\geqslant 0$. Then the total transportation cost per time unit can be computed as

$$\int_X V(x)\psi(x)\,\mathrm{d}x \quad \text{where } V(x_0)=\int_0^\infty l(x(t))\,\mathrm{d}t.$$

Alternatively, one may take the dual viewpoint and compute the stationary density $\rho(x)$ in each point by solving

$$\nabla\cdot(f\rho)=\psi.$$

Then the total cost can be computed as

$$\int_X \rho(x)l(x)\,\mathrm{d}t.$$

Under appropriate assumptions on the boundary of X, equality between the two expressions for the total cost follows from Gauss' theorem:

$$\begin{aligned}\int_X [V\psi-\rho l]\,\mathrm{d}x &= \int_X [V(\nabla\cdot(f\rho))+\nabla V\cdot f\rho]\,\mathrm{d}x\\ &= \int_X \nabla\cdot(Vf\rho)\,\mathrm{d}x = 0.\end{aligned}$$

5. Convexity in nonlinear stabilization

An important application area for Lyapunov function is the synthesis of stabilizing feedback controllers. For a given system, the set of Lyapunov functions is convex. This fact is the basis for many numerical methods, most notably in computation of quadratic Lyapunov functions using linear matrix inequalities [4]. However, when the control law and Lyapunov function are to be found simultaneously, no such convexity property is at hand. In fact, the following example suggested by Praly and Prieur [19,20] shows that the set of (V,u) satisfying

$$0>\nabla V\cdot(f+gu)$$

may not even be connected.

Example 5. Every continuous stabilizing control law $u(x)$ for the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=f(x,u)=\begin{bmatrix}[u(x)-3x_1+3x_2](x_2)^2/|x|^2\\ u(x)\end{bmatrix}$$

must have the property that $u(x)$ has constant sign along the half-line $x_1>0$, $x_2=0$. The reason is that a zero crossing would create a second equilibrium. A strictly decreasing Lyapunov function satisfies

$$0>\nabla V\cdot f(x,u)=\frac{\partial V}{\partial x_2}u(x) \quad \text{for } x_1>0,\ x_2=0$$

so also $\partial V/\partial x_2$ must have constant non-zero sign along the same half line.

The control law $u_l(x)=x_1-2x_2$ is stabilizing with strictly decreasing Lyapunov function $V_l(x)=x_1^2+x_2^2-x_1x_2$. Apparently $\partial V_l/\partial x_2$ is negative along the half-line.

Similarly, the control law $u_g(x)=-3x_1-6x_2$ is stabilizing with Lyapunov function $V_g(x)=x_1^2+x_2^2+x_1x_2$, with $\partial V_g/\partial x_2$ positive along the half line.

In particular, we see that the two control Lyapunov functions V_l and V_g cannot be connected by a continuous path without violating the sign constraint on $\partial V/\partial x_2$.

Given this negative example, it is most striking to find that the corresponding synthesis problem for the new convergence criterion is convex. In fact, the divergence condition

$$0<\nabla\cdot[(f+gu)\rho]$$

is convex in the pair $(\rho,u\rho)$.

To see how this can be used, let us return to the previous example.

Example 5 (*Continued*). For the control laws

$$u_l(x)=x_1-2x_2,\quad u_g(x)=-3x_1-6x_2$$

and sufficiently large $\alpha>0$, the conditions of Theorem 1 are satisfied with the density functions

$$\rho_l(x)=(x_1^2+x_2^2-x_1x_2)^{-\alpha},$$

$$\rho_g(x)=(x_1^2+x_2^2+x_1x_2)^{-\alpha},$$

respectively. The same conditions are satisfied for the convex combination $\rho=(\rho_l+\rho_g)/2$ with $u(x)$ defined by $u\rho=(u_l\rho_l+u_g\rho_g)/2$, i.e.

$$u(x)=\frac{\rho_l(x)u_l(x)+\rho_g(x)u_g(x)}{\rho_l(x)+\rho_g(x)}.$$

Recall that not all convex combinations of ρ_l and ρ_g can correspond to controllers that are globally stabilizing in the sense of Lyapunov. However, they do give rise to systems such that $\lim_{t\to\infty}|x(t)| = 0$ for almost all initial conditions.

6. Proof and generalization

So far, we only proved the second statement of Theorem 1. The first statement is slightly more involved, since we do not know a priori that $\int_Z \rho\,dx < \infty$ for Z defined by (2).

To prove the first statement, it is convenient to start with a more abstract result, which can be viewed as a discrete-time counterpart to Theorem 1. The statement is related to the Poincaré recurrence theorem and makes no reference to topology [18].

Theorem 2. *Consider a measure space $(X, \mathscr{A}, \mu)$, a set $P \subset X$ of finite measure and a measurable map $T: X \to X$. Suppose that*

$$\mu(T^{-1}Y) \leqslant \mu(Y) \quad \textit{for all measurable } Y \subset X. \tag{5}$$

Define Z as the set of elements $x \in P$ such that $T^n(x) \in P$ for infinitely many integers $n \geqslant 0$. Then $\mu(T^{-1}Z) = \mu(Z)$.

Proof. Note that $Z = P \cap (\bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} T^{-k}(P))$, so Z is measurable. Let the superscript "c" denote the complementary set with respect to X, like $Z^c = X \setminus Z$. Define for $n = 1, 2, \ldots$

$$Z_n = \bigcup_{k=1}^{n} T^{-k}(Z), \quad Z_0 = \emptyset.$$

The set Z_n for $n \geqslant 1$ consists of those elements of X that are mapped into Z in n or less steps. Let us prove by induction over n that

$$\mu(T^{-1}(Z)) \geqslant \mu(Z_n \cap Z) + \mu(T^{-n-1}(Z) \cap Z_n^c). \tag{6}$$

The inequality holds trivially for $n = 0$. Assuming that it holds for some $n \geqslant 0$, we get

$$\begin{aligned}
\mu(T^{-1}(Z)) &\geqslant \mu(Z_n \cap Z) + \mu(T^{-n-1}(Z) \cap Z_n^c) \\
&= \mu(Z_n \cap Z) + \mu(T^{-n-1}(Z) \cap Z_n^c \cap Z) \\
&\quad + \mu(T^{-n-1}(Z) \cap Z_n^c \cap Z^c) \\
&\geqslant \mu((Z_n \cup (T^{-n-1}(Z) \cap Z_n^c)) \cap Z) \\
&\quad + \mu(T^{-1}(T^{-n-1}(Z) \cap Z_n^c \cap Z^c)) \\
&= \mu(Z_{n+1} \cap Z) + \mu(T^{-n-2}(Z) \cap Z_{n+1}^c).
\end{aligned}$$

Induction over n therefore proves (6) for all integers $n \geqslant 0$. It follows that

$$\mu(Z) \geqslant \mu(T^{-1}(Z)) \geqslant \sup_n \mu(Z_n \cap Z) = \mu(Z),$$

where the last equality is due to the fact that $Z = (\bigcup_{n=1}^{\infty} T^{-n}(Z)) \cap P$. □

Proof of Theorem 1 (*First statement*). As in the proof of the second statement, we may assume without restriction that ρ is integrable for $|x| \geqslant 1$ and $|f(x)|/|x|$ is bounded by some constant C so $\phi_t(x_0)$ is well defined for all x_0, t. Define $X = \mathbb{R}^n$, $P = \{x \in \mathbb{R}^n : |x| > r\}$, $T(x) = \phi_1(x)$ and

$$\mu(Y) = \int_Y \rho(x)\,dx \quad \text{for measurable } Y \subset X.$$

The condition (5) holds by Lemma A.1. Hence $\mu(T^{-1}Z) = \mu(Z)$ for Z defined as in Theorem 2, so by Lemma A.1 with $t = -1$ and $D = \{x \in \mathbb{R}^n : |x| > \varepsilon\}$ for some sufficiently small $\varepsilon > 0$

$$\int_{-1}^{0} \int_{\phi_\tau(Z)} [\nabla \cdot (f\rho)](x)\,dx\,d\tau = 0.$$

This gives that the Lebesgue measure of $\phi_\tau(Z)$ is zero for almost all $\tau \in [-1, 0]$. Hence, Z must have measure zero and for almost all $x \in P$ there exists $j > 0$ such that

$$|\phi_n(x)| \leqslant r \quad \text{for } n > j.$$

The choice of r was arbitrary, so $\lim_{n\to\infty} |\phi_n(x_0)| = 0$ as $n = 1, 2, \ldots$ for almost all x_0. For a real positive number t, let $[t]$ denote its integer part. The global bound $|f(x)|/|x| < C$ gives $|\dot{x}| < C|x|$ so

$$|x(t)| \leqslant e^{C|t-[t]|}|x([t])| \leqslant e^C |x([t])| \to 0 \quad \text{as } t \to \infty.$$

Hence $|\phi_t(x_0)| \to 0$ also for non-integer values of t and the proof is complete. □

Numerous other convergence criteria can be derived from Theorem 2. To exemplify, we give the following criterion for convergence to infinity in non-autonomous systems.

Corollary 1. *Given $f \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$, let $\dot{x}(t) = f(x, t)$, $x(t_0) = x_0$ have no solutions with finite escape time. Suppose that $\rho \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ is non-negative and*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (f\rho) > 0 \quad \textit{for almost all } x, t. \tag{7}$$

Then $\lim_{t\to\infty} |x(t)| \to \infty$ for almost all x_0, t_0.

Proof. Let $\phi_t(x_0, t_0)$ be the solution of $\dot{x}(t) = f(x,t)$, $x(t_0) = x_0$. Define $X = \mathbb{R}^n \times \mathbb{R}$, $P = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}: |x| < r\}$, $T(x,q) = (\phi_1(x,q), q+1)$ and

$$\mu(Y) = \int_Y \rho(x,q)\,\mathrm{d}x\,\mathrm{d}q \quad \text{for measurable } Y \subset X.$$

Applying Lemma A.1 to the extended system $(\dot{x}, \dot{q}) = (f(x), 1)$ gives (5). Hence, it follows as in the proof of Theorem 1 that $\lim_{t\to\infty} |\phi_t(x)| = \infty$. □

7. Concluding remarks

A new approach to asymptotic analysis for nonlinear systems has been introduced. The new criterion differs from Lyapunov's theorem in several important respects and allows for new applications. In particular, it applies to examples where the system is not globally stable in the sense of Lyapunov.

Another important difference is a convexity property that appears in synthesis of stabilizing control laws. This convexity property is identical to the one that has been exploited for optimal control problems [26,24].

The convergence criterion for differential equations was proved based on an general result, Theorem 2, stated in terms of abstract measures.

In spite of the differences, many extensions to Lyapunov's theorem have analogs in terms of density functions. This includes convergence criteria for non-autonomous systems, inverse theorems and criteria for convergence to invariant sets. We hope to return to some of these issues in later publications.

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Appendix A. A supporting lemma

The proof of the main theorem relies on the following lemma, which can be viewed as a version of Liouville's theorem [2,18].

Lemma A.1. *Let $f \in \mathbb{C}^1(D, \mathbb{R}^n)$ where $D \subset \mathbb{R}^n$ is open and let $\rho \in \mathbb{C}^1(D, \mathbb{R})$ be integrable. For $x_0 \in \mathbb{R}^n$, let $\phi_t(x_0)$ be the solution $x(t)$ of $\dot{x} = f(x)$, $x(0) = x_0$. For a measurable set Z, assume that $\phi_\tau(Z) = \{\phi_\tau(x) \,|\, x \in Z\}$ is a subset of D for all τ between 0 and t. Then*

$$\int_{\phi_t(Z)} \rho(x)\,\mathrm{d}x - \int_Z \rho(z)\,\mathrm{d}z = \int_0^t \int_{\phi_\tau(Z)} [\nabla \cdot (f\rho)](x)\,\mathrm{d}x\,\mathrm{d}\tau.$$

Proof. Note that for every $\mathbb{C}^1$ matrix function $M(t)$ with $M(0) = I$

$$\frac{\partial}{\partial t} \det M(t)\bigg|_{t=0} = \operatorname{trace} M'(0).$$

This follows by direct expansion of the determinant, since the first-order terms in t correspond to the diagonal elements of $M(t)$.

Let $M(t) = (\partial\phi_t/\partial z)(z)$ and use $|\cdot|$ to denote determinant. The differentiability of f gives that $\phi_t(z)$ is of class $\mathbb{C}^1$ in z and $\mathbb{C}^2$ in t [16, p. 40]. Hence

$$\left[\frac{\partial}{\partial t}\left|\frac{\partial \phi_t}{\partial z}(z)\right|\right]_{t=0} = \left[\operatorname{trace} \frac{\partial^2}{\partial t \partial z}\phi_t(z)\right]_{t=0} = \operatorname{trace}\frac{\partial f}{\partial z}(z) = [\nabla \cdot f](z)$$

and with the notation $\rho_t(z) = \rho(\phi_t(z))|(\partial\phi_t/\partial z)(z)|$

$$\frac{\partial}{\partial t}\rho_t(z)\bigg|_{t=0} = \nabla\rho \cdot f + \rho(\nabla \cdot f) = [\nabla \cdot (f\rho)](z),$$

$$\frac{\partial}{\partial t}\rho_t(z)\bigg|_{t=\tau} = \frac{\partial}{\partial h}\left\{\rho_h(\phi_\tau(z))\left|\frac{\partial \phi_\tau}{\partial z}(z)\right|\right\}\bigg|_{h=0} = [\nabla \cdot (f\rho)](\phi_\tau(z))\left|\frac{\partial \phi_\tau}{\partial z}(z)\right|.$$

Let $\chi(\cdot)$ be the characteristic function of Z. Then

$$\begin{aligned}
&\int_{\phi_t(Z)} \rho(x)\,\mathrm{d}x - \int_Z \rho(z)\,\mathrm{d}z \\
&= \int_{\mathbb{R}^n} \rho(x)\chi(\phi_t^{-1}(x))\,\mathrm{d}x - \int_Z \rho(z)\,\mathrm{d}z \\
&= \int_{\mathbb{R}^n} \rho(\phi_t(z))\chi(z)\left|\frac{\partial\phi_t(z)}{\partial z}\right|\mathrm{d}z - \int_Z \rho(z)\,\mathrm{d}z \\
&= \int_Z [\rho_t(z) - \rho(z)]\,\mathrm{d}z \\
&= \int_Z \int_0^t [\nabla \cdot (f\rho)](\phi_\tau(z))\left|\frac{\partial\phi_\tau}{\partial z}(z)\right|\mathrm{d}\tau\,\mathrm{d}z \\
&= \int_0^t \int_{\phi_\tau(Z)} [\nabla \cdot (f\rho)](x)\,\mathrm{d}x\,\mathrm{d}\tau. \qquad \square
\end{aligned}$$

References

[1] A.A. Andronov, E.A. Leontovich, I.I. Gordon, A.G. Maier, Qualitative Theory of Second-Order Dynamic Systems, Wiley, New York, 1973 (Russian edition, 1966).

[2] V.I. Arnold, Mathematical Methods of Classical Mechanics, Graduate Texts in Mathematics, 2nd ed., Springer, Berlin, 1989.

[3] Z. Artstein, Stabilization with relaxed controls, Nonlinear Anal. TMA 7 (1983) 1163–1173.

[4] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, Studies in Applied Mathematics, Vol. 15, SIAM, Philadelphia, 1994.

[5] L.K. Ford, D.K. Fulkerson, Flows in Networks, Princeton University Press, Princeton, NJ, 1962.

[6] W. Hahn, Theory and Applications of Lyapunov's Direct Method, Prentice-Hall, Englewood Cliffs, NJ, 1963.

[7] F.L. Hitchcock, The distribution of a product from several sources to numerous localities, J. Math. Phys. 20 (1941) 224–230.

[8] A. Isidori, Nonlinear Control Systems, Springer, London, 1995.

[9] R.E. Kalman, J.E. Bertram, Control system design via the "second method" of Lyapunov, parts i and ii, J. Basic Eng. 82 (1960) 371–400.

[10] L.V. Kantorovich, On a problem of Monge, Uspekhi Mat. Nauk. 3 (1948) 225–226.

[11] H.K. Khalil, Nonlinear Systems, 2nd ed., Prentice-Hall, Upper Saddle River, NJ, 1996.

[12] P.V. Krasovskii, Stability of Motion, Stanford University Press, Stanford, 1963.

[13] M. Krstic, I. Kanellakopoulos, P. Kokotovich, Nonlinear and Adaptive Control Design, Wiley, New York, 1995.

[14] Y.S. Ledyaev, E.D. Sontag, A Lyapunov characterization of robust stabilization, Nonlinear Anal. 37 (1999) 813–840.

[15] S. Lefschetz, J.P. La Salle, Stability of Lyapunov's Direct Method, Academic Press, New York, 1961.

[16] S. Lefschetz, Differential Equations: Geometric Theory, Dover Publications, New York, 1977.

[17] M.Y. Li, J.S. Muldowney, On r.a. smith's autonomous convergence theorem, Rocky Mountain J. Math. 25 (1) (1995) 365–379.

[18] R. Mane, Ergodic Theory and Differentiable Dynamics, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer, Berlin, English edition, 1987.

[19] L. Praly, personal communication.

[20] C. Prieur, L. Praly, Uniting local and global controllers, in: Proceedings of IEEE Conference on Decision and Control, AZ, December 1999, pp. 1214–1219.

[21] S. Rachev, L. Rüschendorf, Mass Transportation Problems, Volume I: Theory. Probability and its Applications, Springer, New York, 1998.

[22] N. Rouche, P. Habets, M. Laloy, Stability Theory by Lyapunov's Direct Method, Springer, New York, 1977.

[23] R.A. Smith, Some applications of Haussdorff dimension inequalities for ordinary differential equations, Proc. Roy. Soc. Edinburgh 164A (1986) 235–259.

[24] R. Vinter, Convex duality and nonlinear optimal control, SIAM J. Control Optim. 31 (2) (1993) 518–538.

[25] J.C. Willems, Dissipative dynamical systems, part I: General theory; part II: Linear systems with quadratic supply rates, Arch. Rational Mech. Anal. 45 (5) (1972) 321–393.

[26] L.C. Young, Lectures on the Calculus of Variations and Optimal Control Theory, W.B. Saunders, Philadelphia, PA, 1969.