Doing Quantum Mechanics with Control Theory

H. H. Rosenbrock

Abstract—Bellman showed how classical mechanics can be obtained from Hamilton’s principle by dynamic programming. If we add noise in a particular way, we obtain Schrödinger’s equation and many other results from Hamilton’s principle by dynamic programming. If we add noise in a particular way, we obtain Schrödinger’s equation and many other results from Hamilton’s principle by dynamic programming.

Index Terms—Dynamic programming, Hamilton’s principle, quantum mechanics.

I. HAMILTON’S PRINCIPLE EXTENDED

Hamilton’s principle, as presented for example by Lanczos [1], gives a concise and elegant basis for classical mechanics, which has had a strong influence on later developments. There has been no similar basis for quantum mechanics, and the work described briefly here, which has been done over the past 15 years [2]–[5], has been aimed at filling this gap. The brief account describes some of the results that have been obtained and their possible significance for the philosophy of science. The notation is described in the Appendix, and for simplicity the application to a single particle in one dimension is described.

It was noted many years ago [6] that Hamilton’s principle falls within the scope of dynamic programming. If \( H(x, p, t) \) is the Hamiltonian of a particle with position \( x \) and momentum \( p \) moving on the real line, Hamilton’s principle can be stated in the form

\[
\delta \int_{v, p} \left[ pv - H(x, p, t) \right] d\tau = 0, \quad v = dx/d\tau
\]

(1)

where as a result of the optimization \( v \) and \( p \) become functions of the remaining variables \( x \) and \( t \) on an optimal trajectory. Writing \( W(x, t) \) for the optimal (stationary) value of the integral, which we regard as a “cost,” Bellman’s equation is

\[
\text{stat}_{v, p} \left[ \frac{dW}{dt} + pv - H \right] = \text{stat}_{v, p} \left[ \frac{\partial W}{\partial t} + v \frac{\partial W}{\partial x} + pv - H \right] = 0
\]

(2)

giving

\[
p = -\frac{\partial W}{\partial x}, \quad v = \frac{\partial H}{\partial p}, \quad \dot{H} = \frac{\partial W}{\partial t}
\]

(3)

and from the first of these equations

\[
\frac{dp}{dt} = -\left[ \frac{\partial^2 W}{\partial x \partial t} + v \frac{\partial^2 W}{\partial x^2} \right] = -\left[ \frac{\partial H}{\partial x} - \frac{\partial H}{\partial p} \right].
\]

(4)

In the last equation, \( H \) is a function of \( x, t \), and the optimal \( p(x, t) = -\frac{\partial W}{\partial x} \), so that the quantity in square brackets is \( \partial H / \partial x \) with \( p \) held constant. Thus if in \( H(x, p, t) \), \( x \) and \( p \) are regarded as independent variables, (3) and (4) give Hamilton’s equations

\[
\frac{dx}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x}.
\]

(5)

The above procedure develops a closed-loop solution of (1), but in the absence of random disturbances this is the same as an open-loop solution.

In quantum mechanics we deal with complex functions such as \( \psi \), and with probability distributions of variables. To introduce these features we replace \( x \) by the complex variable \( q \), and \( v = dx/d\tau \) in (1) by \( dq = v d\tau + i \hbar \frac{\partial}{\partial q} \), where \( \hbar \) is a generalized Wiener process (random walk, or “integrated white noise”), and \( n \) is complex, \( n^2 = -i \hbar /m \). With \( \mathcal{E} \) denoting the expectation, (1) is then replaced by

\[
\delta \mathcal{E} \int_t^T [\dot{\psi} - \mathcal{H}(q, \psi, \tau)] d\tau = 0
\]

(6)

where variables such as \( \dot{\psi}, \psi \), \( \mathcal{H} \) become complex, this being indicated where necessary by the tilde. We seek a closed-loop solution of (6), which is now distinct from an open-loop solution, and obtain it as before by dynamic programming.

Bellman’s equation (2) is now replaced [7] by

\[
\text{stat}_{\psi, \dot{\psi}} \left[ \frac{\partial W}{\partial t} + \dot{\psi} \frac{\partial W}{\partial q} + \frac{n^2}{2} \frac{\partial^2 W}{\partial q^2} + \psi \dot{\psi} - \mathcal{H} \right] = 0
\]

(7)

where we have used

\[
\mathcal{E} d\tau = 0, \quad \mathcal{E} dt \psi^2 = dt, \quad \mathcal{E} dq^2 = \mathcal{E}(\dot{q}^2 dt^2 + 2 \dot{q} dt n dz + n^2 dz^2) = n^2 dt
\]

(8)

to first order in \( dt \), and (3) now becomes

\[
\dot{\psi} = -\frac{\partial \mathcal{H}}{\partial q}, \quad \dot{\psi} = \frac{\partial \mathcal{H}}{\partial p}, \quad \mathcal{H} = \frac{\partial W}{\partial t} - i \hbar \frac{\partial^2 W}{2m \partial q^2}
\]

(9)

where the extended Hamiltonian \( \mathcal{H} \) replaces \( H \) in (3). The sum total of the changes therefore is to make the variables in (3) complex, and to add a second-order term in the last equation.

We still seem far from quantum mechanics, but a simple change of variables takes us home. Put \( \mathcal{W} = i \hbar \log \psi \) in (9) and write \( \dot{\psi} = -i \hbar \partial q / \partial q \) to give

\[
\dot{\psi} = -\frac{\partial \mathcal{W}}{\partial q} \psi^{-1} \dot{\psi}, \quad i \hbar \frac{\partial^2 \mathcal{W}}{\partial q^2} = \psi^{-1} \dot{\psi}^2 \psi - \dot{\psi}^2
\]

(10)

and then define \( \mathcal{H} = \frac{\partial \mathcal{W}}{\partial t} \) to obtain

\[
\mathcal{H} \psi = i \hbar \frac{\partial \mathcal{W}}{\partial t} = \left[ \frac{\dot{\psi}^2}{2m} + \dot{\psi} \right] \psi = \mathcal{H} \psi
\]

(11)

which is Schrödinger’s equation, extended from the real line to the complex plane designated by \( q \). The bracketed quantity in (11) is obtained by replacing \( \dot{\psi} \) in \( \mathcal{H} \) by the operator \( \dot{\psi} = -i \hbar \partial q / \partial q \).

We have therefore embedded the orthodox theory, of complex functions on a real space, in a theory of complex functions on a complex space. The variable \( q \) can no longer give the position of the particle, and we say that it defines the position of a “complex image.” To revert to the orthodox theory we make the following postulate: If the expected value of some property (such as momentum \( \dot{\psi} \) or energy \( \mathcal{H} \)) of the complex images in an ensemble has the real value \( \alpha \) on the real axis, then \( \alpha \) is the value of the corresponding property of the physical particle.

Thus if \( \alpha \) is a real constant and we have \( \mathcal{H} \psi = \psi^{-1} \mathcal{H} \psi = \alpha \), then \( \mathcal{H} \psi = \alpha \psi \), which can be satisfied only if \( \alpha \) is an eigenvalue of the operator \( \mathcal{H} \) and \( \psi \) is a corresponding eigenvector. Similarly for the momentum, \( \dot{\psi} = \psi^{-1} \dot{\psi} \psi = \alpha \) gives \( \dot{\psi} \psi = \alpha \psi \), and we can follow the
same procedure for $x$ if we introduce an operator $\hat{x}$ by $x = \psi^{-1} \hat{x} \psi = \alpha$, giving $\hat{x} \psi = \alpha \psi$, for which the eigenfunctions are $\psi = \delta(x - \alpha)$. In the above development $W$ and $\dot{W}$ have the opposite sign to the one usually given by physicists. This is because dynamic programming operates backward in time, so that if in (2) and (7) is at the lower limit of integration.

A result which has not been given above is that the density of physical particles on the real axis is $\rho = \psi^* \psi$ and that $\rho$ satisfies the equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \quad (12)$$

where $\alpha$ is the real part of $\hat{\alpha}$. The proof of this, though straightforward, is rather long and is therefore omitted; it has been given elsewhere [3], [4], and the foregoing development for a single particle has been generalized and extended to special relativity [3], [4].

From (6), and from a single postulate, we can in this way [3]–[5] obtain a number of results which are usually given as separate postulates in the elementary development of quantum mechanics.

a) Momentum $\dot{\rho}$ in classical mechanics is to be replaced on the real axis by the operator $\hat{\rho} = -i\hbar \partial /\partial x$.

b) The function $\exp(W / i\hbar)$ is the wave function $\psi$, satisfying Schrödinger’s equation.

c) The density of particles in an ensemble on the real axis is $\rho = \psi^* \psi$.

d) The density $\rho$ satisfies (12), where $u$ is $\text{Re} \hat{v}$, which is $\text{Re}(\psi^* \hat{p} \psi / m)$ for a single particle.

e) When $\psi$ corresponds to a real quantity such as $\rho$ or $\hat{\rho}$, it must be an eigenfunction of the corresponding operator $\hat{\rho}$, $\hat{\rho}$, etc. The corresponding eigenvalue is the value of the variable. With c) this gives the usual operator calculus.

f) When a system is to be considered as partly classical and partly quantum mechanical, a particular symmetrical combination of the variables is to be chosen.

g) The Poisson bracket of classical mechanics is to be replaced in quantum mechanics by the commutator.

II. EQUIVALENT THEORIES

There is a development in the philosophy [8] and sociology [9] of science which regards theory as underdetermined by facts. Any number of theories, it is suggested, may explain a given set of data: “First, scientists will choose theories which accord with the facts, and then, among those that are not thereby eliminated, they will choose the simplest theory” [9].

Simplicity is clearly not the only criterion of choice that may be used. A theory will be highly regarded if it suggests new phenomena which are then substantiated by experiment. Again, in any age there may be an accepted dogma, and theories will not be accepted without a struggle if they do not agree with it, as we see in Cardinal Bellarmine’s letter to Foscarni [10]. It is one thing, he said, to demonstrate that astronomical observations could be explained by assuming that the earth went round the sun, but this “is not the same thing as to demonstrate that in fact the sun is at the center and the earth in the heavens. I believe that the first demonstration may exist, but I have very grave doubts about the second; and in case of doubt one may not abandon the Holy Scriptures.”

Bellarmine was right in believing that the heliocentric and earth-centered theories were equivalent at that time, because both were equally able to account for all the observational knowledge that was available. He rejected the heliocentric theory on grounds of theological dogma. Equivalence, of course, can be destroyed by further data.

Regarded as an example of underdetermination, Hamilton’s principle (1) is an alternative theory explaining the same facts as Hamilton’s or Lagrange’s equations. The extended principle (6) is similarly an alternative theory accounting for a number of facts in elementary quantum mechanics and available for further development. It has an advantage in simplicity because (6) and its single accompanying postulate are equivalent to Schrödinger’s equation and its numerous accompanying postulates. Some limitations on the equivalence of (6) and (11) are considered below.

A. The Use of $W$

Quantum mechanics is so highly developed and so successful that it will be hard for any alternative formulation to suggest new phenomena not also predicted by the standard theory. Nevertheless there are some results which, although in principle they are obtainable equally from (11) or from Schrödinger’s equation, are more easily found from the former. Schrödinger’s equation leads to a theory in terms of $\psi$, and has the benefit of linearity. The extended Hamilton’s principle (6) leads to a theory in terms of $W$, which has a close relationship with the classical theory. In consequence, there are a number of results which are more easily available in terms of $W$ than of $\psi$.

(i) First, if we regard $n$ as a free parameter and replace $ih$ by $-i \hbar m^2$, (6) gives rise to (9) for all $n$, zero or nonzero, real or complex. For nonzero $n$, the substitution $W = -i m^2 \log \psi$ then linearizes the third equation in (9), whether $n$ is real or complex. Particular systems have been intensively studied: the zero gives classical mechanics, the nonzero gives classical systems with diffusion, $n^2 = -i \hbar /m$ gives quantum mechanics. However, the general system described by (6) and (9) for all $n$ deserves study which it has not yet received.

(ii) Second, if the boundary conditions in (1) are $\{x, t\}$ and $\{x, t')$, we can indicate the dependence of $W$ on these conditions by writing it $W(x, t; x', t')$. If $\partial^2 W / \partial x^2$ is not a function of $x$, it is then easy to show from (9) that the kernel (propagator, Green’s function) [11] for the corresponding quantum mechanical system is

$$K(x, t; x', t') = \alpha \exp \left\{ \left[ W + \frac{i \hbar}{2m} \int \frac{\partial^2 W}{\partial x^2} \ dt \right] /i\hbar \right\} \quad (13)$$

where $\alpha$ is a normalizing constant. This is an interesting alternative to the van Vleck formula [12] which is quite different in form.

(iii) Third, if an electromagnetic plane wave traveling the direction $x_3$ is quantized by the procedure following from (6), we find [3] from $W$ that when the energy is $(k + 1/2) \hbar \omega$, the electric field is proportional to

$$\left[ i \mu \omega_a - i \hbar \sum_{k=1}^k 1 / (a - \beta_k) \right] \exp(i \omega x_3 /c) \quad (14)$$

with $\mu$ magnitude of a component $A_1$ of the vector potential; $k$ constant; $\beta_k$ $k$ zeros of the Hermite polynomial $H_k(a \sqrt{\mu \omega /\hbar})$.

The optimal trajectories defined by (14) when $k = 2$ are sketched in Fig. 1. Here the availability of a picture in complex space provides insight.

At $P$ and $Q$ the electric field is zero, whereas at $\beta_1$ and $\beta_2$ it becomes infinite. There are as many points $\beta_i$ as there are photons associated with the electric field. The addition of each new point $\beta$ corresponds to an increase in energy by $\omega \hbar$. The points $\beta$ travel in the direction of $x_3$ with speed $c$. As $k$ increases, the density of the zeros of $H_k$ on the $x_3$ axis approximates to the classical energy density.
The development from (6) showed how a particle could become associated in quantum mechanics with a wave function \( \psi \). The further development here suggests how a quantum mechanical field can have singularities with many of the properties of photons. How far this tentative identification of photons with singularities of the electric field can be carried is a matter for further investigation. Neither spin nor quantum field theory have so far been investigated by methods based upon (6).

### B. Results Anterior to (11)

In Section II-A, results were given which are suggested by, or more easily obtainable from, a theory based upon \( \hat{W} \) than one based upon \( \psi \). The first of these theories arises most naturally from (6), and the second from (11), but the two are interconvertible by the relation \( \hat{W} = i\hbar \log \psi \), and to that extent are equivalent theories. There are, however, results which cannot be obtained from (11) because they result from a “substructure” provided by (6). Most of the consequences of this substructure are reflected in Schrödinger’s equation, but not all.

(i) As a first example we have, on using (8) as in (7),

\[
\mathcal{E}d\hat{p} = \left[ \frac{\partial \hat{p}}{\partial t} + \hat{v} \frac{\partial \hat{p}}{\partial q} - \frac{i\hbar}{2m} \frac{\partial^2 \hat{p}}{\partial q^2} \right] dt \tag{15}
\]

which on using (9) gives

\[
\mathcal{E}d\hat{p} = \left[ \frac{\partial^2 \hat{W}}{\partial q \partial t} + \frac{\partial \hat{H}}{\partial \hat{p}} \frac{\partial \hat{p}}{\partial q} + \frac{i\hbar}{2m} \frac{\partial^2 \hat{W}}{\partial q^2} \right] dt
= \left[ - \frac{\partial}{\partial q} \left( \hat{H} + \frac{i\hbar}{2m} \frac{\partial^2 \hat{W}}{\partial q^2} \right) \right. \frac{\partial \hat{p}}{\partial \hat{p}} + \frac{i\hbar}{2m} \frac{\partial^3 \hat{W}}{\partial q^3} \left. \right] dt
= - \left[ \frac{\partial \hat{H}}{\partial q} - \frac{\partial \hat{H}}{\partial \hat{p}} \frac{\partial \hat{p}}{\partial q} \right] dt. \tag{16}
\]

As in (4) we can interpret (16) and the second of equations (9), in a slightly unorthodox notation, as

\[
\mathcal{E}d\hat{q} = \frac{\partial \hat{H}}{\partial \hat{p}}. \quad \mathcal{E}d\hat{p} = - \frac{\partial \hat{H}}{\partial q} \tag{17}
\]

when \( \hat{q} \) and \( \hat{p} \) are regarded as independent variables. Equation (17) is a pleasing result, showing how Hamilton’s equations generalize in quantum mechanics.

(ii) Second, Bellman’s functional equation, when generalized to complex variables, gives the following discrete-time approximation to the solution of (6):

\[
\hat{W}_r(q, t_r) = \lim_{n \to \infty} \sum_{t_r-1}^{t_{r+1}} \{ \hat{L}(q, \hat{v}, t_r)(t_r-1, t_r) + \hat{W}_{r-1}(q, \hat{v}, t_r-1, t_r) + n z, t_{r-1} \} P(z, t_{r-1} - t_r, t_r) dz \tag{18}
\]

where the probability density \( P \) is

\[
P(z, t_{r-1} - t_r) \equiv [2\pi(t_r - t_{r-1})]^{-1/2} \exp\left[ -z^2/2(t_r - t_{r-1}) \right]. \tag{19}
\]

Here (Fig. 2) \( t_r \) denotes a time between the initial time \( t = t_N \) and the final time \( t_f = t_0 \), the numbering being backward in time.

An iteration can be started from a given function \( \hat{W}_0 \) of \( q_0 \) at \( t_0 \), giving \( \hat{v}_1 \) and \( \hat{W}_1 \) as functions of \( q_1 \) at \( t_1 \), and can be carried back to give the functions \( \hat{v}_1 \) and \( \hat{W}_N \) at the initial time \( t_N \). If we wish the boundary condition at the final time to be a specified point \( Q \), we can use a penalty function \( \hat{W}_0(q_0, t_0) \) to achieve this, or more simply can set the velocity \( \hat{v}_1 \) at \( t_1 \) equal to \( (Q - q_1)/(t_0 - t_1) \). This last procedure introduces a well-known difficulty, because \( \hat{W}_0(q_0, t_0) \) becomes infinite, but the function \( \hat{W}_N(q_N, t_N) \) can nevertheless be found. All of this would involve some heavy computing if it were carried out numerically, but algebraically it is more amenable.

Some details of this calculation resemble those of Feynman’s path integral method [11], but the two procedures are distinct. Path integrals rely on the linearity of Schrödinger’s equation. The dynamic programming procedure has no need for linearity, but depends instead on the fact that causal influence acts only in the forward direction of time. Thus (18) has to be solved backward from the final time, whereas path integrals can be calculated in either direction.

(iii) Third, in the standard theory it has always been difficult to account for interference, as exemplified [11] by the two-slit experiment. Electrons, for example, appear to behave like waves when they pass through slits in a screen, but like particles when they are detected at a second screen. The development from (6) offers a simple explanation in terms of the ensemble over which the expectation is taken. When it is known through which slit, \( A \) or \( B \), a particle has passed, the ensemble consists simply of particles which have passed through that slit, and there is a resulting wave function \( \psi_A \) or \( \psi_B \). When this information is not available, the entity “particle having passed through \( A \)” is in-
distinguishable from the entity “particle having passed through B.” The two therefore cannot be associated with different functions \( \psi_A, \psi_B \), but only with a single ensemble having a single \( \psi' \), which is easily seen to be \( \psi' = \psi_A + \psi_B \).

Thus when a detector is put into operation at one of the slits, the information that can be used to determine the optimal \( \dot{v} \) at points between the two screens is increased, and it is this which accounts for the change in behavior of the system. A version of (6) with real \( n \) has (regrettably) been used [13] to obtain an optimal control algorithm for a swarm of cruise missiles subject to random disturbances. These, like the electrons, exhibit interference, which can be demonstrated by computer simulation, but it is cosh-like rather than cos-like.

The explanation given here is available when we start from (6), but is not available in the standard theory when we start from (11) because that is evaluated on the assumption that the initial position of a particle is available at all times, together with the current position, to determine \( \dot{v} \). By introducing the anterior stage, from (6) to (11), into the theory we thus provide room for explanations which are otherwise unavailable.

(iv) Fourth, in (6) and in Bellman’s functional equation (18), the direction of time is involved in an essential way. The evolution in time of the random disturbance in (6) cannot be reversed, while the validity of (18) depends on the direction of causal influence in time. In Schrödinger’s equation these considerations have been lost.

### III. Causality and Purpose

The closed-loop solution (7) of (6) represents a goal-seeking behavior, as is most easily seen from (18) when the initial point \( q \) and final point \( Q \) are given. If it arose from a technical system, this could represent the following problem. A body is to move from a position \( q \) at the initial time \( t \) to \( Q \) at the final time \( t_f \), and in doing so it is to minimize the expected “cost” \( W(q, t) \). The position of the body is to be measured at intermediate times \( t \), and a new optimal velocity will be calculated from the measured \( q \) at \( t \), so taking account of perturbations of the trajectory due to random disturbances. Here “minimize” has been used instead of “make stationary,” but the latter condition is the one that would actually be used. The language of “cost,” “measuring \( q_r \),” etc., are entirely inappropriate for the physical system. Nevertheless, the fact is that Schrödinger’s equation arises from a closed-loop solution of (6), comparable to the technical control requirement.

Now there is a multitude of quotations condemning the notion of goal-seeking (or teleology or purpose) in scientific descriptions of nature, and contrasting it with causality (or objectivity). The rejection of purpose is presented as a prior requirement for the scientist: “The cornerstone of the scientific method is… the systematic denial that ‘true’ knowledge can be reached by interpreting phenomena in terms of purpose… For science the only a priori is the postulate of objectivity” [14], and “To declare in earnest that there is some deeper meaning attached to the idea of nature’s acting in such a way that it may make some quantity a minimum [i.e., satisfies Hamilton’s principle] is perhaps a relic from the time when the universe was believed to be driven by a superhuman being” [15], and “For the scientist there is only ‘being,’ but no wishing, no valuing, no good, no evil, no goal” [16].

This can be understood as a reaction to the conflict in the seventeenth century between the Church and a nascent science, but to anyone acquainted with control theory it must seem outdated. A human purpose can be incorporated in a machine, since any feasible purpose has a policy that will accomplish it, and the policy is a set of causal relations: differential equations and the like. The machine is built to obey the causal relations, and in doing so it fulfills the purpose. One does not credit the machine with intelligence or foresight, but to fully understand the machine, one needs to know the purpose as well as the policy.

In the same way, (6) defines a purpose, and (11) is the policy that will achieve it, subject to necessary postulates and elaborations in both cases. For science it should not matter which we use, so long as the equivalence between the two persists. The rejection of purpose in favor of policy can only be regarded as dogma: a decision made before any explanation of the phenomena is sought.

One of the aims of the work described above was to show that dogmatic rejection of purpose can impoverish science: explanations of phenomena may be available from the purposive description which cannot be obtained from the causal description. There are, however, much wider consequences of the rejection of purpose, for example in the philosophy of science and for our relation to nature and to society.

As an example, living organisms exhibit clear purposes, and if the substrate of quantum mechanical particles from which life evolves is described as purposeless the questions arises “how can purpose arise from a purposeless substrate?” With somber regret [14], or with proselytizing enthusiasm [17], the answer given is that natural selection, acting upon chance variations, produces this effect. If we describe quantum mechanics as goal-seeking we have a different situation. We do not need to say that the argument for natural selection acting upon a purposeless substrate is wrong: but we can say that it answers the wrong question. If the substrate were purposeless things might indeed happen in the way suggested, but with a goal-seeking substrate a different question needs to be answered: “how can a new purpose arise in living organisms, from a substrate which already has purpose, given that a system cannot simultaneously seek two goals unless these are in a special relationship with each other?”

More broadly still, we cannot act in the world without acquiring a moral responsibility, and scientific investigation is an action like any other. If science describes the world as devoid of purpose then, because inert matter commands no respect, scientists as well as technologists are in danger of rejecting their moral responsibility. Ample evidence of the way in which this can happen has been given elsewhere [18].

### Appendix

### Notations

The symbols and definitions given here are those that apply in the theory derived from (6). Those accompanied by * do not appear in the standard treatment of quantum mechanics. Those which do appear in the standard treatment (for example \( \tilde{p} \)) are shown as functions of \( q \) arising from (6), though in the standard theory they appear as functions of the real variable \( x \).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>( \mathcal{E} )</td>
<td>( \hbar /2\pi )</td>
<td>expectation taken over an ensemble where ( \hbar ) is Planck’s constant</td>
</tr>
<tr>
<td>( H(x, p, t) )</td>
<td>( \partial W / \partial t )</td>
<td>classical Hamiltonian expression for the energy, in which ( p ) and ( x ) are independent variables; but when derived from ( W ) on an optimal trajectory ( p ) becomes a function of ( x ) and ( t )</td>
</tr>
<tr>
<td>( \tilde{H}(q, \tilde{p}, t) )</td>
<td>as in (9)</td>
<td>obtained by substituting ( q, \tilde{p} ) for ( x, p ) in ( H(x, p, t) ); not equal to the energy</td>
</tr>
</tbody>
</table>
\[ *\frac{\partial}{\partial t} (q, \dot{p}, t) \quad \partial W / \partial t \quad \text{q.m. energy} \]

\[ \ddot{H}(q, \dot{p}, t) \quad \text{as in (11)} \quad \text{q.m. energy operator, obtained by substituting } q, \dot{p} \text{ for } x, p \text{ in } H(x, p, t) \]

\[ L(x, v, t) \quad p v - H \quad \text{classical Lagrangian} \]

\[ *L(q, \dot{v}, t) \quad \ddot{v} - \ddot{H} \quad \text{q.m. Lagrangian} \]

\[ p = -\partial W / \partial x \quad \text{classical momentum} \]

\[ *\dot{p} = -\partial \dot{W} / \partial q \quad \text{q.m. momentum} \]

\[ \dot{p} = -i \hbar \partial / \partial q \quad \text{q.m. momentum operator} \]

\[ v = \partial H / \partial p \quad \text{classical velocity} \]

\[ *\dot{v} = \partial \ddot{H} / \partial \dot{p} \quad \text{q.m. velocity} \]

\[ V(x, t) \quad \text{classical potential energy} \]

\[ *V(q, t) \quad \text{obtained by substituting } q \text{ for } x \text{ in } V(x, t) \]

\[ W \quad \text{as in (1)} \quad \text{classical action integral with reversed sign} \]

\[ *W \quad \text{as in (6)} \quad \text{q.m. action integral} \]

\[ \rho \quad \text{density of physical particles on the real axis} \]

\[ \psi = \exp(W / i\hbar) \quad \text{wave function satisfying Schrödinger’s equation} \]

\[ \text{stat}_{v, p, q} \quad \text{stationary value obtained by varying the functions } v, p \]

\[ \text{stat}_{\dot{v}, \dot{p}, q} \quad \text{stationary value obtained by varying the functions } \dot{v}, \dot{p} \]

\[ \delta_{v, p} \quad \text{variation of the succeeding expression with respect to } v, p \]

\[ \delta_{v, \dot{p}} \quad \text{variation of the succeeding expression with respect to } \dot{v}, \dot{p} \]

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**Robust \( H_\infty \) Control of Uncertain Markovian Jump Systems with Time-Delay**

Yong-Yan Cao and James Lam

**Abstract**—This correspondence is concerned with the robust stochastic stabilizability and robust \( H_\infty \) disturbance attenuation for a class of uncertain linear systems with time delay and randomly jumping parameters. The transition of the jumping parameters is governed by a finite-state Markov process. Sufficient conditions on the existence of a robust stochastic stabilizing and suboptimal \( H_\infty \) state-feedback controller are presented using the Lyapunov functional approach. It is shown that a robust stochastically stabilizing \( H_\infty \) state-feedback controller can be constructed through the numerical solution of a set of coupled linear matrix inequalities.

**Index Terms**—Jumping parameters, linear matrix inequality (LMI), linear uncertain systems, robust control, time-delay systems.

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**I. INTRODUCTION**

A great deal of attention has recently been devoted to the Markovian jump linear systems. This family of systems is modeled by a set of linear systems with the transitions between the models determined by a Markov chain taking values in a finite set. It was introduced by Krasovskii and Lidskii in 1961 [13] and may represent a large variety of processes, including those in production systems and economic problems. Developments in control engineering regarding applications, stability conditions, and optimal control problems for jump linear systems are reported in [1], [3], [8]–[10], [12], and [17]. On the other hand, time-delay systems have been studied extensively on the subject of stability and control over the years; see [4], [5], and [16] for instance. The problem of robust \( H_\infty \) control of linear uncertain systems with time delay has gathered much attention, and some sufficient conditions have been presented [6], [7], [11], [15].

In this correspondence, we study the robust stochastic stabilizability and robust \( H_\infty \) disturbance attenuation for a class of uncertain linear

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**REFERENCES**


